The Smith Chart

Smith chart was developed by P.H. Smith in 1939 prior to the age of computers! However, it has been used widely since then as a graphical technique for designing transmission line circuits.

Parametric equations:

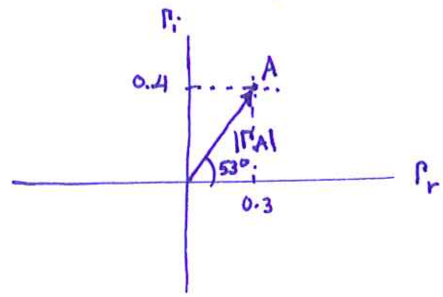
Reflection coefficient Γ is complex in general:

$$\Gamma = |\Gamma| e^{j\theta_r} = \Gamma_r + j\Gamma_i$$

$$\Gamma_r = |\Gamma| \cos\theta_r$$

$$\Gamma_i = |\Gamma| \sin\theta_r$$

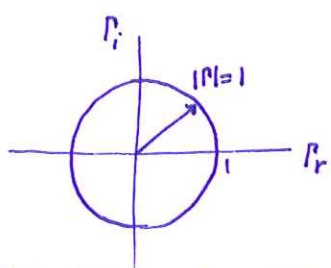
The smith chart lies in the complex plane of Γ. For example $\Gamma_A = 0.3 + j0.4$ is shown as:



$$|\Gamma_A| = (0.3^2 + 0.4^2)^{1/2} = 0.5$$

$$\theta_r = \tan^{-1}\frac{0.4}{0.3} = 53^\circ$$

$|\Gamma| = 1$ corresponds to a unit circle:



Since $|\Gamma| \le 1$ any Γ is within this circle. Note that never $|\Gamma| > 1$.

We can also show the impedance on the smith chart. For this we always normalize the impedance Z_L by the charactristic impedance Z_0:

$$z_L = \frac{Z_L}{Z_0}$$

We will show z on the smith chart.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{z_L - 1}{z_L + 1} \rightarrow \boxed{z_L = \frac{1+\Gamma}{1-\Gamma}}$$ normalized load impedance.

z_L is complex in general: $z_L = r_L + jx_L$

$$z_L = r_L + jx_L = \frac{1+\Gamma}{1-\Gamma} = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i} = \frac{(1+\Gamma_r + j\Gamma_i)(1-\Gamma_r + j\Gamma_i)}{(1-\Gamma_r - j\Gamma_j)(1-\Gamma_r + j\Gamma_j)}$$

$$= \frac{1-\Gamma_r^2 - \Gamma_i^2 + j(\Gamma_i - \Gamma_i\Gamma_r + \Gamma_i + \Gamma_i\Gamma_r)}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$= \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2} + j\frac{2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2}$$

$$r_L = \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2} \qquad x_L = \frac{2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2}$$

If r_L is given, we can write:

$$r_L = \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2} \rightarrow 1-\Gamma_r^2-\Gamma_i^2 = r_L(1-\Gamma_r)^2 + r_L\Gamma_i^2 = r_L + r_L\Gamma_r^2 - 2r_L\Gamma_r + r_L\Gamma_i^2$$

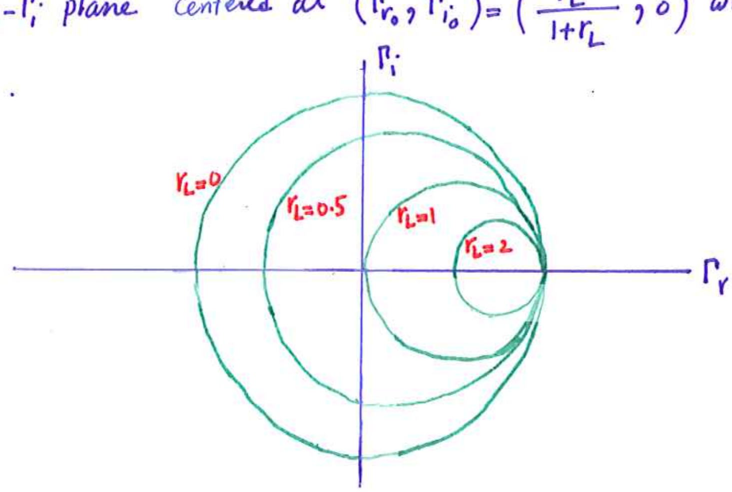
$$\rightarrow 1-r_L = (1+r_L)\Gamma_r^2 - 2r_L\Gamma_r + (1+r_L)\Gamma_i^2$$

$$\frac{1-r_L}{1+r_L} = \Gamma_r^2 - \frac{2r_L}{1+r_L}\Gamma_r + \frac{r_L^2}{(1+r_L)^2} - \frac{r_L^2}{(1+r_L)^2} + \Gamma_i^2$$

$$\frac{1-r_L}{1+r_L} + \frac{r_L^2}{(1+r_L)^2} = \left(\Gamma_r - \frac{r_L}{1+r_L}\right)^2 + \Gamma_i^2$$

$$\frac{1-r_L^2+r_L^2}{(1+r_L)^2} = \left(\Gamma_r - \frac{r_L}{1+r_L}\right)^2 + \Gamma_i^2 \quad \Rightarrow \quad \left(\Gamma_r - \frac{r_L}{1+r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r_L}\right)^2$$

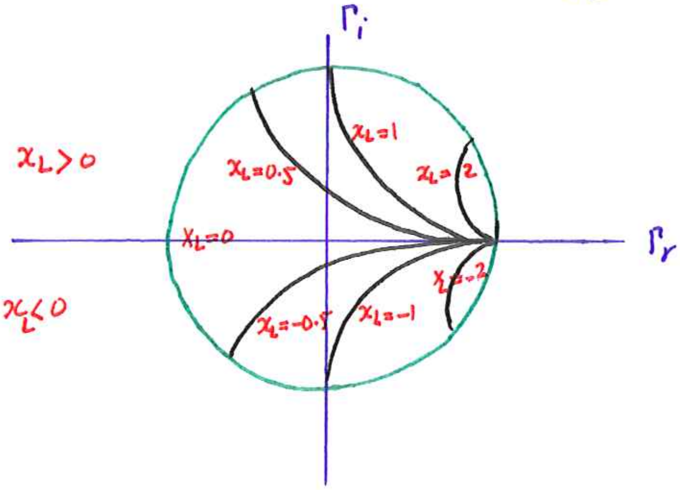
This is a circle in Γ_r-Γ_i plane centered at $(\Gamma_{r_0}, \Gamma_{i_0}) = \left(\frac{r_L}{1+r_L}, 0\right)$ with radius of $\frac{1}{1+r_L}$.



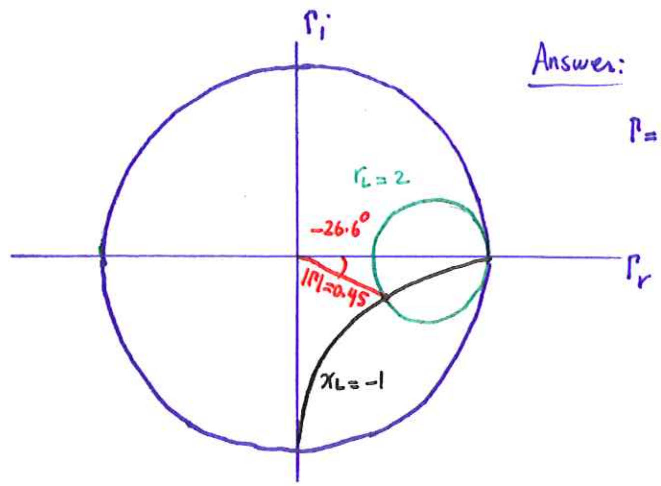
A similar examination for x_L yields:

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2$$

This is a circle centered at $(\Gamma_{r_0}, \Gamma_{i_0}) = (1, \frac{1}{x_L})$ with radius $\frac{1}{x_L}$.



<u>Example</u> Find the corresponding Γ for a normalized load impedance of $z_L = 2 - j1$.



<u>Answer:</u>

$$\Gamma = |\Gamma| e^{j\theta_r} = 0.45 e^{-j26.6^\circ}$$

<u>Input Impedance:</u>

We have: $$Z_{in} = Z_0 \frac{1 + \Gamma e^{-j2\beta l}}{1 - \Gamma e^{-j2\beta l}} \xrightarrow[\text{to } Z_0]{\text{normalized}} z_{in} = \frac{Z_{in}}{Z_0} = \frac{1 + \Gamma e^{-j2\beta l}}{1 - \Gamma e^{-j2\beta l}}$$

Γ is the reflection at load: $\Gamma = |\Gamma| e^{j\theta_r}$ as before.

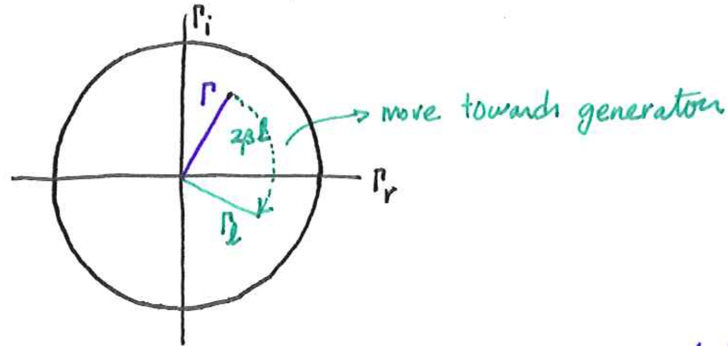
If we define: $\Gamma_l \triangleq \Gamma e^{-j2\beta l} = |\Gamma| e^{j\theta_r} e^{-j2\beta l} = |\Gamma| e^{j(\theta_r - 2\beta l)}$

we call this as <u>phase shifted</u> voltage reflection coefficient.

(Note: Compare with $z_L = \frac{1+\Gamma}{1-\Gamma}$, Γ is replaced with $\Gamma e^{-j2\beta l}$ which is phase shifted)

$$z_{in} = \frac{1+\Gamma_\ell}{1-\Gamma_\ell} \quad \text{and} \quad z_L = \frac{1+\Gamma}{1-\Gamma} \quad \text{where } \Gamma_\ell = \Gamma e^{-j2\beta\ell}$$

In smith chart, phase shift of $e^{-j2\beta\ell}$ is rotation of θ_r by $2\beta\ell$ in clockwise direction.



One complete rotation takes the point back to itself. In length this is equal to:

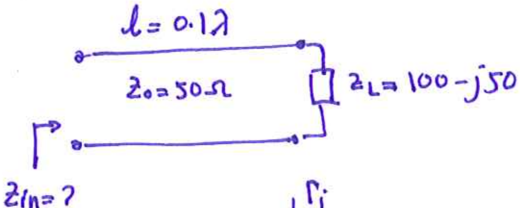
$$2\beta\ell = 2\pi \rightarrow \ell = \frac{\pi}{\beta} = \frac{\pi}{\frac{2\pi}{\lambda}} = \frac{\lambda}{2}$$

So $\ell = \frac{\lambda}{2}$ has no effect as we saw before. The outermost circle in a smith chart has a scale called the wavelength toward generator (WTG) scale.

If we move towards load, the rotation is counterclockwise. There is also a scale around smith chart for moving towards load which is called wavelength toward load (WTL) scale.

Example

Find Z_{in} in this line:



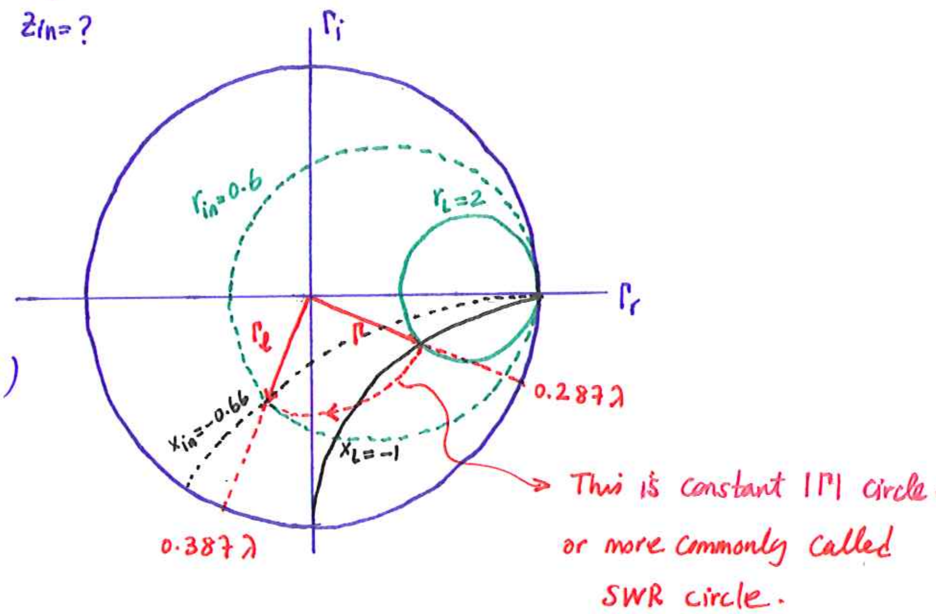
$$z_L = \frac{100-j50}{50} = 2-j1$$

Answer:

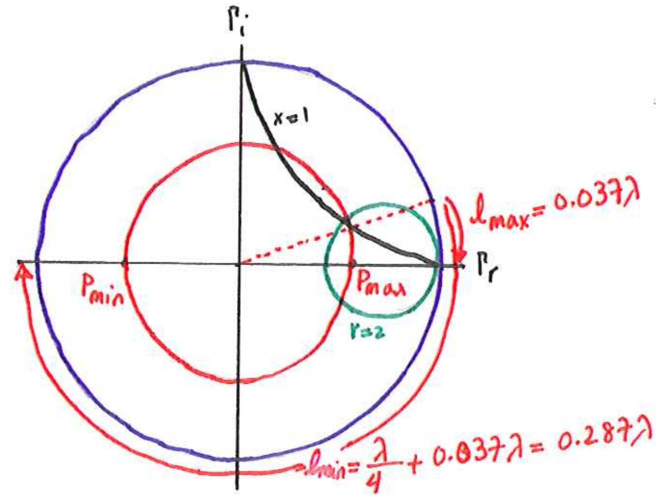
$z_{in} = 0.6 - j0.66$

$Z_{in} = Z_0 z_{in} = (50)(0.6-j0.66)$

$Z_{in} = 30 - j33 \ \Omega$



SWR, Voltage Maxima, and Minima



Consider a load $z_L = 2 + j1$

The SWR circle intersects the real axis Γ_r at two points: P_{min} and P_{max}.

At both points $\Gamma_i = 0$ and $\Gamma = \Gamma_r$. Also on the real axis $x_L = 0$. we have:

$$\Gamma = \frac{z_L - 1}{z_L + 1} \quad \text{at } P_{min} \text{ and } P_{max}: \ \Gamma = \Gamma_r = \frac{r_L - 1}{r_L + 1} \rightarrow |\Gamma| = \Gamma_r$$

P_{min} corresponds to the case $r_L < 1$

P_{max} " " $r_L > 1$

we also had $S = \dfrac{1 + |\Gamma|}{1 - |\Gamma|} \Rightarrow S = \dfrac{1 + \Gamma_r}{1 - \Gamma_r} \rightarrow \Gamma_r = \dfrac{S - 1}{S + 1}$

$S = r_L$ at P_{max}

$S = r_L$ only at P_{max} since $S \geq 1$. We can conclude that S is equal to the value of r_L at P_{max}, the point at which the SWR circle intersects the real Γ axis on the right-hand side of the chart's center.

Point P_{max} and P_{min} also represent the distances from the load the the voltage is minimum and maximum, respectively:

- At P_{max}, total phase of Γ is $0 \Rightarrow$ max voltage
- At P_{min}, total phase of Γ is $\pi \Rightarrow$ min voltage

(note: total phase of Γ is $\theta_r - 2\beta l$).

From the figure, the first voltage maximum happens at $l = 0.037\lambda$ from the load and first minimum happens at $l = 0.287\lambda$ from the load.

(Note: V_{max} corresponds to I_{min} and V_{min} corresponds to I_{max}.)

Impedance to Admittance Transformer

It is sometimes easier to work with admittance than with impedance:

admittance $Y = \frac{1}{Z}$

$$Z = R + jX \rightarrow Y = \frac{1}{R+jX} = \frac{R-jX}{R^2+X^2} = G + jB$$

$$\rightarrow G = \frac{R}{R^2+X^2} \quad \text{and} \quad B = \frac{-X}{R^2+X^2}$$ the unit is S.

normalized admittance is $y = \frac{Y}{Y_0} = \frac{G}{Y_0} + j\frac{B}{Y_0} = g + jb$ where $Y_0 = \frac{1}{Z_0}$ characteristic admittance.

$g = \frac{G}{Y_0} = GZ_0$ (dimensionless) $\rightarrow$ normalized conductance

$b = \frac{B}{Y_0} = BZ_0$ (dimensionless) $\rightarrow$ normalized sucseptance

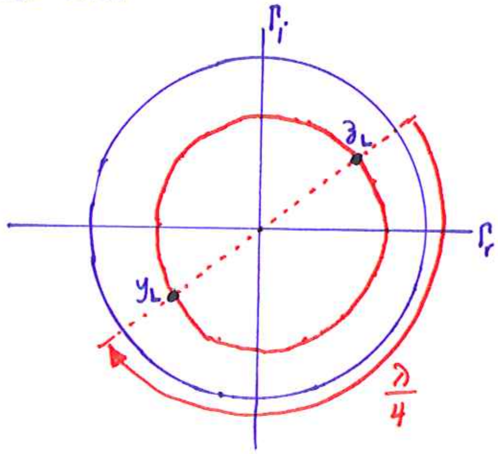
The normalized load admittance y_L is:

$$\boxed{y_L = \frac{1}{z_L} = \frac{1-\Gamma}{1+\Gamma}}$$

Consider normalized input impedance z_{in} at a distance $l = \frac{\lambda}{4}$ from the load:

$$2\beta l = \frac{4\pi}{\lambda}\frac{\lambda}{4} = \pi \Rightarrow z_{in}(l = \tfrac{\lambda}{4}) = \frac{1+\Gamma e^{-j\pi}}{1-\Gamma e^{-j\pi}} = \frac{1-\Gamma}{1+\Gamma} = y_L$$

Thus, rotation by $\frac{\lambda}{4}$ on the smith chart transforms z_L to y_L. This is the opposite point on the SWR circle:



$z_L = \frac{1}{y_L}$

rotate by $\frac{\lambda}{4}$ to get y_L. This is the opposite point on SWR circle.

Example

a – find voltage reflection coefficient

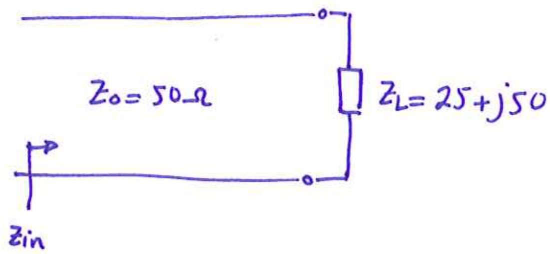
b – the voltage standing wave ratio

c – distance from the first voltage maximum

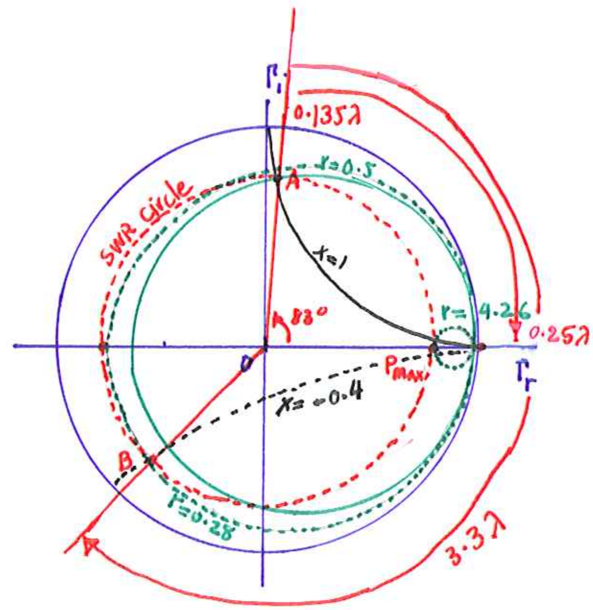
d – distance from the first voltage minimum

e – input impedance of the line at $l = 3.3\lambda$

f – input admittance of the line



$z_L = \frac{Z_L}{Z_0} = 0.5 + j1$



Answer: (a) From point A: $\theta_r = 83^\circ$

Measure OA with ruler and the radius of the circle (R) $\Rightarrow |\Gamma| = \frac{OA}{R} = 0.62$

$\rightarrow \Gamma = 0.62 e^{j83^\circ}$

(b) Plot SWR circle. It crosses Γ_r at P_{max}. r_L at this point is VSWR: $S = r_L = 4.26$

(c) $0.25\lambda - 0.135\lambda = 0.115\lambda \rightarrow l_{max} = 0.115\lambda$

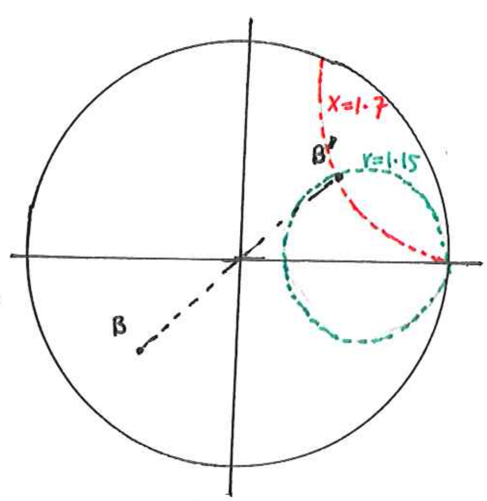
(d) $l_{min} = \frac{\lambda}{4} + l_{max} = 0.25\lambda + 0.115\lambda = 0.365\lambda$

(e) Corresponds to point B $\rightarrow z_{in} = 0.28 - j0.4 \rightarrow Z_{in} = z_{in} Z_0 = 14 - j20\ \Omega$

(f) y_{in} is the opposite point to B:

B′ corresponds to $y_{in} \rightarrow$

$y_{in} = 1.15 + j1.7$



$\rightarrow Y_{in} = y_{in} Y_0 = \frac{1.15 + j1.7}{Z_0 = 50} = 0.023 + j0.034\ (S)$

Curl of a Vector Field

$$\vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{y}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{z}\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right)$$

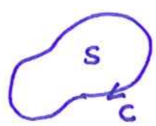
Vector Identities

$$\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B}$$

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ for any vector $\vec{A}$

$\vec{\nabla}\times(\vec{\nabla}V) = 0$ for any scalar V

Stokes's Theorem



$$\boxed{\int_S (\nabla\times B)\cdot ds = \oint_C B\cdot dl}$$

if $\nabla\times B = 0$, the field B is said to be conservative or irrotational, because its circulation as in the above equation is zero.

Laplacian Operator

$$\nabla^2 \triangleq \nabla\cdot\nabla = \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}\right)$$

so that $\boxed{\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla}V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$ when V is scalar.

$$\boxed{\nabla^2\vec{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{E} = \hat{x}\,\nabla^2 E_x + \hat{y}\,\nabla^2 E_y + \hat{z}\,\nabla^2 E_z}$$

It can be shown: $\boxed{\nabla^2\vec{E} = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{E})}$

Note: $\nabla^2 E_x = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2}$

$\nabla^2 E_y = \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2}$

$\nabla^2 E_z = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2}$

Laplacian of a scalar is a scalar.

Laplacian of a vector is a vector.